

# Colombeau Algebra: A pedagogical introduction

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## Abstract

A simple pedagogical introduction to the Colombeau algebra of generalised functions is presented, leading the standard definition.

## 1 Introduction

This is a pedagogical introduction to the Colombeau algebra of generalised functions. I will limit myself to the Colombeau Algebra over  $\mathbb{R}$ . Rather than  $\mathbb{R}^n$ . This is mainly for clarity. Once the general idea has been understood the extension to  $\mathbb{R}^n$  is not too difficult. In addition I have limited the introduction to  $\mathbb{R}$  valued generalised functions. To replace with  $\mathbb{C}$  valued generalised functions is also rather trivial.

I hope that this guide is useful in your understanding of Colombeau Algebras. Please feel free to contact me.

There is much general literature on Colombeau Algebras but I found the books by Colombeau himself[1] and the Masters thesis by Tạ Ngọc Trí[2] useful.

## 2 Test functions and Distributions

The set of infinitely differentiable functions on  $\mathbb{R}$  is given by

$$\mathcal{F}(\mathbb{R}) = \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi^{(n)} \text{ exists for all } n\} \quad (1)$$

Test functions are those function which in addition to being smooth are zero outside an interval, i.e.

$$\mathcal{F}_0(\mathbb{R}) = \{\phi \in \mathcal{F}(\mathbb{R}) \mid \text{there exists } a, b \in \mathbb{R} \text{ such that } \phi(x) = 0 \text{ for } x < a \text{ and } x > b\} \quad (2)$$

I will assume the reader is familiar with distributions, either in the notation of integrals or as linear functionals. Thus the most important distributions is the Dirac- $\delta$ . This is defined either as a “function”  $\delta(x)$  such that

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0) \quad (3)$$

Or as a distribution  $\Delta : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathbb{R}$ ,

$$\Delta[\phi] = \phi(0) \quad (4)$$

We will refer to (3) as the integral notation and (4) as the Schwartz notation. An arbitrary distribution will be written either as  $\psi(x)$  for the integral notation or  $\Psi$  for the Schwartz notation.

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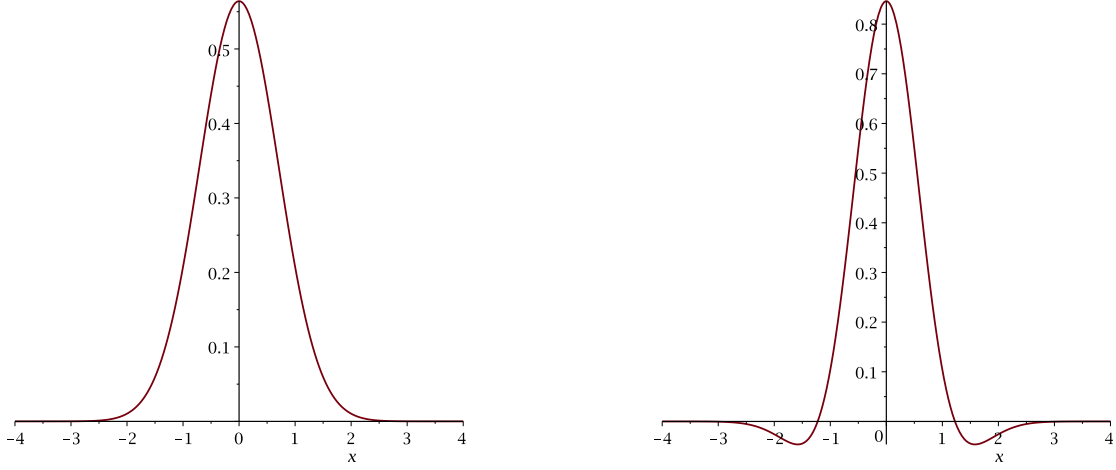


Figure 1: Plots of  $\phi_1 \in \mathbb{A}_1$  and  $\phi_3 \in \mathbb{A}_3$

### 3 Function valued distributions

The first step in understanding the Colombeau Algebra is to convert distributions into a new object which takes a test functions  $\phi$  and gives a functions

$$\mathbf{A} : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$$

This is achieved by using translation of the test functions. Given  $\phi \in \mathcal{F}_0$  then let

$$\phi^y \in \mathcal{F}_0(\mathbb{R}), \quad \phi^y(x) = \phi(x - y) \quad (5)$$

Then in integral notation

$$\overline{\psi}[\phi](y) = \int_{-\infty}^{\infty} \psi(x) \phi(x - y) dx \quad (6)$$

and in Schwartz notation

$$\overline{\Psi}[\phi](y) = \Psi[\phi^y] \quad (7)$$

We will define the Colombeau Algebra in such a way that they include the elements  $\overline{\psi}$  and  $\overline{\Psi}$ . The overline will be used to covert distributions into elements of the Colombeau algebra.

We label the set of all function valued functionals

$$\mathcal{H}(\mathbb{R}) = \{\mathbf{A} : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})\} \quad (8)$$

We see below that we need to restrict  $\mathcal{H}(\mathbb{R})$  further in order to define the Colombeau algebra  $\mathcal{G}(\mathbb{R})$ .

Observe that we use a slightly non standard notation. Here  $\mathbf{A}[\phi] : \mathbb{R} \rightarrow \mathbb{R}$  is a function, so that given a point  $x \in \mathbb{R}$  then  $\mathbf{A}[\phi](x) \in \mathbb{R}$ . One can equally write  $\mathbf{A}[\phi](x) = \mathbf{A}(\phi, x)$ , which is the standard notation in the literature. However I claim that the notation  $\mathbf{A}[\phi](x)$  does have advantages.

### 4 Three special examples.

For the Dirac- $\delta$  we see that

$$\overline{\delta} = \overline{\Delta} = \mathbf{R} \quad (9)$$

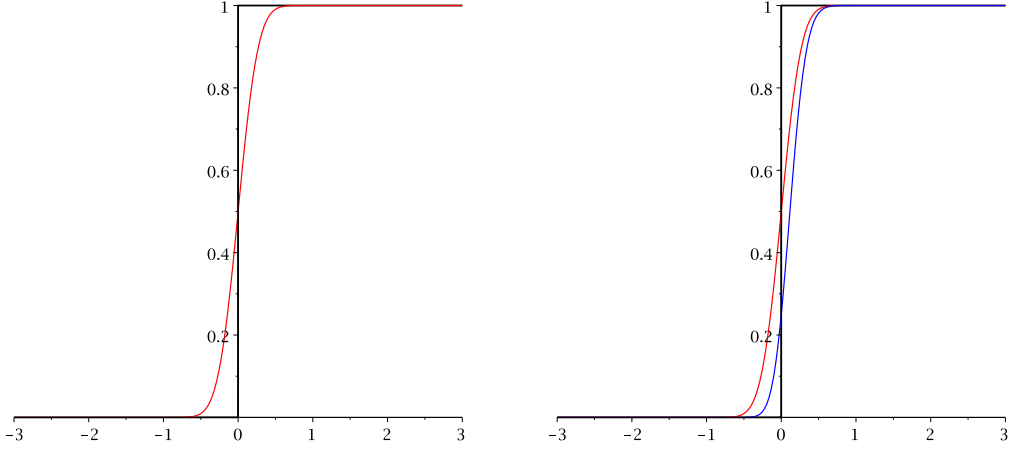


Figure 2: Heaviside (black) and  $\bar{\theta}[\phi]$  (red) and  $(\bar{\theta}[\phi])^2$  (blue)

where  $\mathbf{R} \in \mathcal{H}(\mathbb{R})$  is the reflection map

$$\mathbf{R}[\phi](y) = \phi(-y) \quad (10)$$

This is because

$$\bar{\delta}[\phi](y) = \int_{-\infty}^{\infty} \delta(x) \phi(x - y) dx = \phi(-y)$$

and is Schwartz notation

$$\bar{\Delta}[\phi](y) = \Delta[\phi^y] = \phi^y(0) = \phi(-y)$$

Regular distribution: Given any function  $f \in \mathcal{F}$  then there is a distribution  $f^D$  given by

$$f^D[\phi] = \int_{-\infty}^{\infty} f(x) \phi(x) dx \quad (11)$$

Thus we set  $\bar{f} = \overline{f^D} \in \mathcal{H}(\mathbb{R})$  as

$$\bar{f}[\phi](y) = f^D[\phi^y] = \int_{-\infty}^{\infty} f(x) \phi(x - y) dx \quad (12)$$

The other important generalised functions are the regular functions. That is given  $f \in \mathcal{F}$  we set

$$\tilde{f} \in \mathcal{H}(\mathbb{R}), \quad \tilde{f}[\phi] = f \quad \text{that is} \quad \tilde{f}[\phi](y) = f(y) \quad (13)$$

The effect of replacing  $\bar{\psi}[\phi_\epsilon]$  is to smooth out  $\psi$ . Examples of  $\phi$  are given in figure 1. The action  $\bar{\theta}[\phi]$  where  $\theta$  is the Heaviside function is given in figure 2.

## 5 Sums and Products

Given two Generalised functions  $\mathbf{A}, \mathbf{B} \in \mathcal{H}(\mathbb{R})$  then we can define there sum and product in the natural way

$$\mathbf{A} + \mathbf{B} \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (\mathbf{A} + \mathbf{B})[\phi] = \mathbf{A}[\phi] + \mathbf{B}[\phi] \quad \text{i.e.} \quad (\mathbf{A} + \mathbf{B})[\phi](y) = \mathbf{A}[\phi](y) + \mathbf{B}[\phi](y) \quad (14)$$

and

$$\mathbf{A}\mathbf{B} \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (\mathbf{A}\mathbf{B})[\phi] = \mathbf{A}[\phi]\mathbf{B}[\phi] \quad \text{i.e.} \quad (\mathbf{A}\mathbf{B})[\phi](y) = \mathbf{A}[\phi](y)\mathbf{B}[\phi](y) \quad (15)$$

We see that the product of delta functions  $\bar{\delta}^2 \in \mathcal{H}(\mathbb{R})$  is clearly defined. That is

$$\bar{\delta}^2[\phi](y) = (\bar{\delta}[\phi]\bar{\delta}[\phi])(y) = \bar{\delta}[\phi](y)\bar{\delta}[\phi](y) = (\phi(-y))^2$$

Although this is a generalised function, it does not correspond to a distribution, via (7). That is there is no distribution  $\Psi$  such that  $\bar{\Psi} = (\bar{\delta})^2$ .

Likewise we can see from figure 2 that  $(\bar{\theta})^2[\phi] = (\bar{\theta}[\phi])^2 \neq \bar{\theta}[\phi]$ .

## 6 Making $\bar{f}$ and $\tilde{f}$ equivalent

Now compare the generalised function  $\bar{f}$  and  $\tilde{f}$  (12),(13). We would like these two generalised functions to be equivalent, so that we can write  $\bar{f} \sim \tilde{f}$ . One of the results of making  $\bar{f} \sim \tilde{f}$  is that if  $f, g \in \mathcal{F}$  then

$$\overline{(fg)} \sim \widetilde{(fg)} = \tilde{f} \tilde{g} \sim \bar{f} \bar{g}$$

In the Colombeau algebra, which is a quotient of equivalent generalised functions, we say that  $\bar{f}$  and  $\tilde{f}$  are the same generalised function.

The goal therefore is to restrict the set of possible  $\phi$  so that when they are acted upon by  $(\bar{f} - \tilde{f})$  the difference is *small*, where *small* will be made technically precise. When we think of quantities being small, we need a 1-parameter family of such quantities such that in the limit the difference vanishes. Here we label the parameter  $\epsilon$  and we are interested in the limit  $\epsilon \rightarrow 0$  from above, i.e. with  $\epsilon > 0$ . Given a one parameter set of functions  $g_\epsilon \in \mathcal{F}$  then one meaning to say  $g_\epsilon$  is small is if  $g_\epsilon(y) \rightarrow 0$  for all  $y$ . However we would like a whole hierarchy of smallness. That is for any  $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  then we can say

$$g_\epsilon = \mathcal{O}(\epsilon^q) \quad (16)$$

if  $\epsilon^{-q}g_\epsilon(y)$  is bounded as  $\epsilon \rightarrow 0$ . Note that we use bounded, rather than tends to zero. However, clearly, if  $g_\epsilon = \mathcal{O}(\epsilon^q)$  then  $\epsilon^{-q+1}g_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

We will also need the notion of  $g_\epsilon = \mathcal{O}(\epsilon^q)$  where  $q < 0$ . Thus we wish to consider functions which blow up as  $\epsilon \rightarrow 0$ , but not too quickly. Such functions will be called *moderate*.

Technically we say  $g_\epsilon$  satisfies (16) if for any interval  $(a, b)$  there exists  $C > 0$  and  $\eta > 0$  such that

$$\epsilon^{-q}|g_\epsilon(x)| < C \quad \text{for all} \quad a \leq y \leq b \quad \text{and} \quad 0 < \epsilon < \eta \quad (17)$$

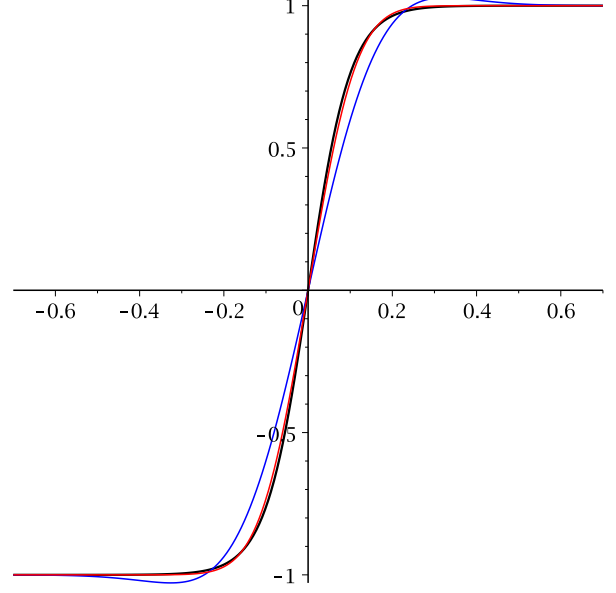
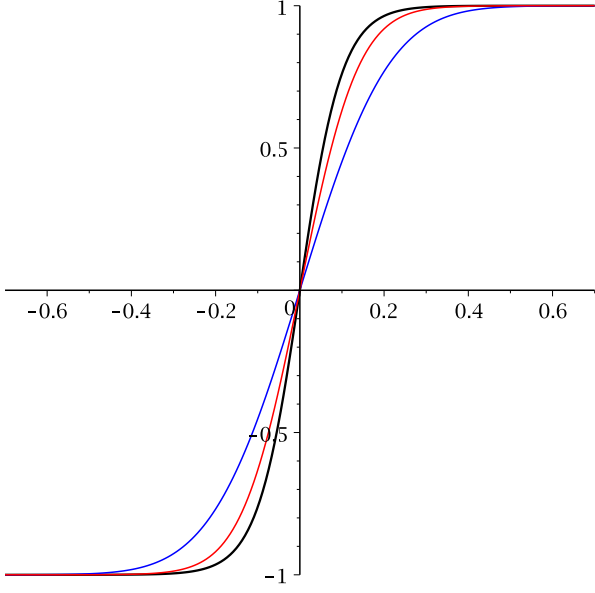
We introduce the parameter  $\epsilon$  via the test functions, replacing  $\phi \in \mathcal{F}_0$  with  $\phi_\epsilon \in \mathcal{F}_0$  where

$$\phi_\epsilon(x) = \frac{1}{\epsilon} \phi\left(\frac{x}{\epsilon}\right) \quad (18)$$

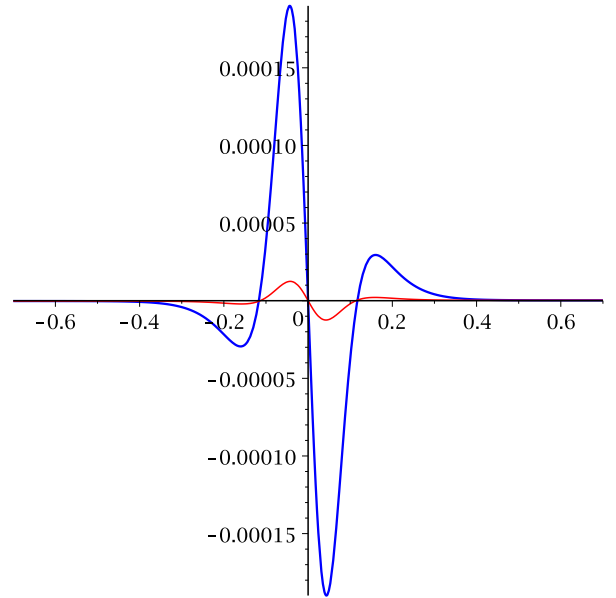
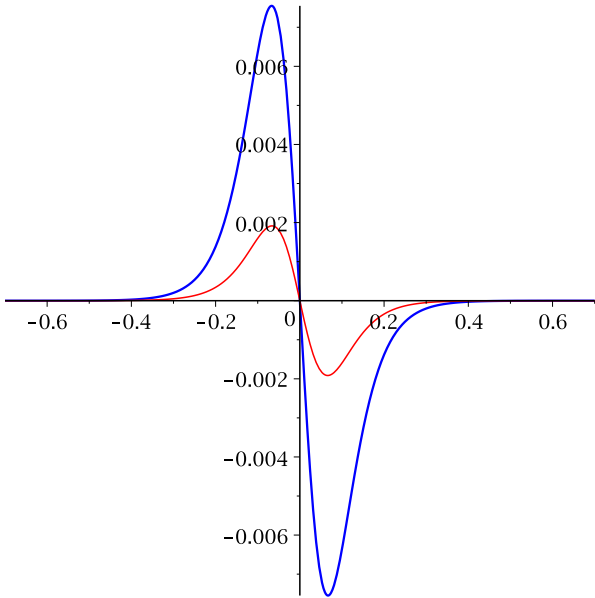
Observe that as  $\epsilon \rightarrow 0$  then  $\phi_\epsilon$  becomes narrower and taller, in a definite sense more like a  $\delta$ -function. Thus we consider a generalised function  $\mathbf{A}$  to be small, if for some appropriate set of test functions  $\phi \in \mathcal{F}_0$  and for some  $q \in \mathbb{Z}$ ,  $\mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^q)$ .

Let us first restrict  $\phi \in \mathcal{F}_0$  to be test function which integrate to 1. That is we define  $\mathbb{A}_0 \subset \mathcal{F}_0$ ,

$$\mathbb{A}_0 = \left\{ \phi \in \mathcal{F}_0 \mid \int_{-\infty}^{\infty} \phi(x) dx = 1 \right\} \quad (19)$$



$f$  (back),  $\bar{f}[\phi_1|_{\epsilon=0.2}]$  (blue) and  $\bar{f}[\phi_1|_{\epsilon=0.1}]$  (red).     $f$  (back),  $\bar{f}[\phi_3|_{\epsilon=0.2}]$  (blue) and  $\bar{f}[\phi_3|_{\epsilon=0.1}]$  (red).



$(\bar{f} - \tilde{f})[\phi_1|_{\epsilon=0.02}]$  (blue) and  
 $(\bar{f} - \tilde{f})[\phi_1|_{\epsilon=0.01}]$  (red).

$(\bar{f} - \tilde{f})[\phi_3|_{\epsilon=0.02}]$  (blue) and  
 $(\bar{f} - \tilde{f})[\phi_3|_{\epsilon=0.01}]$  (red).

Figure 3: Plots of  $\bar{f}[\phi_\epsilon]$  with  $f(x) = \tanh(10x)$

Given  $\phi \in \mathbb{A}_0$  and setting  $z = (x - y)/\epsilon$  so that  $x = y + \epsilon z$

$$\begin{aligned}\bar{f}[\phi_\epsilon](y) &= f^D[\phi_\epsilon^y] = \int_{-\infty}^{\infty} f(x)\phi_\epsilon(x - y)dx = \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x)\phi\left(\frac{x - y}{\epsilon}\right)dx \\ &= \int_{-\infty}^{\infty} f(y + \epsilon z)\phi(z)dz\end{aligned}\tag{20}$$

Thus as  $\epsilon \rightarrow 0$  then  $f(y + \epsilon z) \approx f(y)$  so that, since  $\phi \in \mathbb{A}_0$ ,

$$\bar{f}[\phi_\epsilon](y) = \int_{-\infty}^{\infty} f(y + \epsilon z)\phi(z)dz \approx \int_{-\infty}^{\infty} f(y)\phi(z)dz = f(y) \int_{-\infty}^{\infty} \phi(z)dz = f(y) = \tilde{f}[\phi_\epsilon](y)$$

In fact since  $f(y + \epsilon z) - f(y) = \mathcal{O}(\epsilon)$  we can show using (17) that

$$\text{if } \phi \in \mathbb{A}_0 \quad \text{then} \quad (\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon)\tag{21}$$

This is good so far, but we want to further restrict the set  $\phi$  so that we can satisfy

$$(\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^q)\tag{22}$$

to any order of  $\mathcal{O}(\epsilon^q)$ .

Taylor expanding  $f(y + \epsilon z)$  to order  $\mathcal{O}(\epsilon^{q+1})$  we have

$$f(y + \epsilon z) = \sum_{r=0}^q \frac{\epsilon^r z^r f^{(r)}(y)}{r!} + \mathcal{O}(\epsilon^{q+1})$$

Thus

$$\begin{aligned}(\bar{f} - \tilde{f})[\phi_\epsilon](y) &= \int_{-\infty}^{\infty} (f(y + \epsilon z) - f(y))\phi(z)dz = \int_{-\infty}^{\infty} \left( \sum_{n=1}^q \frac{\epsilon^n z^n f^{(n)}(y)}{n!} + \mathcal{O}(\epsilon^{q+1}) \right) \phi(z)dz \\ &= \sum_{n=1}^q \frac{\epsilon^n f^{(n)}(y)}{n!} \int_{-\infty}^{\infty} z^n \phi(z)dz + \mathcal{O}(\epsilon^{q+1})\end{aligned}\tag{23}$$

Thus we can satisfy (16) to order  $\mathcal{O}(\epsilon^{q+1})$  if the first  $q$  moments of  $\phi(z)$  vanish:

$$\int_{-\infty}^{\infty} z^r \phi(z)dz = 0 \quad \text{for } 1 \leq r \leq q$$

We now define all the elements with vanishing moments.

$$\mathbb{A}_q = \left\{ \phi \in \mathcal{F}_0(\mathbb{R}) \mid \int_{-\infty}^{\infty} \phi(z)dz = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^r \phi(z)dz = 0 \quad \text{for } 1 \leq r \leq q \right\}\tag{24}$$

So clearly  $\mathbb{A}_{q+1} \subset \mathbb{A}_q$ . We can show that these functions exist. Thus from (23) we have

$$\phi \in \mathbb{A}_q \quad \text{implies} \quad (\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+1})\tag{25}$$

Two example test functions  $\phi_1 \in \mathbb{A}_1$  and  $\phi_3 \in \mathbb{A}_3$  are given in figure 1. The result  $\bar{f}[\phi_\epsilon]$ , (12), (20) is given in fig 3.

The easiest way to construct  $\phi \in \mathbb{A}_q$  is to choose a test function  $\psi$  and then set

$$\phi(z) = \lambda_0 \psi(z) + \lambda_1 \psi'(z) + \cdots + \lambda_{q-1} \psi^{(q-1)}(z)$$

where  $\lambda_0, \dots, \lambda_{q-1} \in \mathbb{R}$  are constants determined by (24).

## 7 Null and moderate generalised functions.

As we stated we wanted  $\bar{f}$  and  $\tilde{f}$  to be considered equivalent. From (25) we have  $\phi \in \mathbb{A}_q$  then  $(\bar{f} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+1})$ . We generalise this notion. We say that  $\mathbf{A}, \mathbf{B} \in \mathcal{H}(\mathbb{R})$  are equivalent,  $\mathbf{A} \sim \mathbf{B}$ , if for all  $q \in \mathbb{N}$  there is a  $p \in \mathbb{N}$  such that

$$\phi \in \mathbb{A}_p \quad \text{implies} \quad \mathbf{A}[\phi_\epsilon] - \mathbf{B}[\phi_\epsilon] = \mathcal{O}(\epsilon^q) \quad (26)$$

We label  $\mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{H}(\mathbb{R})$  the set of all elements which are *null*, that is equivalent to the zero element  $\mathbf{0} \in \mathcal{H}(\mathbb{R})$  that is

$$\mathcal{N}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \mathbf{A} \sim \mathbf{0}\}$$

I.e.

$$\mathcal{N}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } p \in \mathbb{N} \text{ there exists } q \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_q, \mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^p)\} \quad (27)$$

Examples of null elements are of course  $\bar{f} - \tilde{f} \in \mathcal{N}^{(0)}(\mathbb{R})$ , which is true by construction. Another example is  $\mathbf{N} \in \mathcal{N}^{(0)}(\mathbb{R})$  which is given by

$$\mathbf{N}[\phi](y) = \phi(1) \quad (28)$$

Since for any  $\phi \in \mathbb{A}_0$  there exists  $\eta > 0$  such that  $1/\eta$  is outside the support of  $\phi$ . Thus  $\phi_\epsilon(1) = 0$  for all  $\epsilon < \eta$  and hence  $\mathbf{N}[\phi_\epsilon] = 0$  so  $\mathbf{N} \in \mathcal{N}^{(0)}(\mathbb{R})$ . However, although  $\mathbf{N} \in \mathcal{N}^{(0)}$ , we can choose  $\phi$  so that  $\mathbf{N}[\phi](y) = \phi(1)$  is any value we choose. Thus knowing that a generalised function  $\mathbf{A}$  is null says nothing about the value of  $\mathbf{A}[\phi]$  but only the limit of  $\mathbf{A}[\phi_\epsilon]$  as  $\epsilon \rightarrow 0$ .

We would like  $\mathcal{N}^{(0)}(\mathbb{R})$  to form an ideal in  $\mathcal{H}(\mathbb{R})$ , that is that if  $\mathbf{A}, \mathbf{B} \in \mathcal{N}^{(0)}(\mathbb{R})$  and  $\mathbf{C} \in \mathcal{H}(\mathbb{R})$  then

- $\mathbf{A} + \mathbf{B} \in \mathcal{N}^{(0)}(\mathbb{R})$  and
- $\mathbf{AC} \in \mathcal{N}^{(0)}(\mathbb{R})$ .

It is easy to see that the first of these is automatically satisfied. However the second requires one additional requirement. We need

$$\mathbf{C}[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N}) \quad (29)$$

for some  $N \in \mathbb{Z}$ . Thus although  $\mathbf{C}[\phi_\epsilon] \rightarrow \infty$  as  $\epsilon \rightarrow 0$  we don't want it to blow up too quickly. Now we have the following:

Given  $\mathbf{A} \in \mathcal{N}^{(0)}(\mathbb{R})$  and  $\mathbf{C}$  satisfying (29) and given  $q \in \mathbb{N}_0$  then there exists  $p \in \mathbb{Z}$  such that  $\phi \in \mathbb{A}_p$  implies  $\mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N})$ . Hence

$$(\mathbf{AC})[\phi_\epsilon] = \mathbf{A}[\phi_\epsilon]\mathbf{C}[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N})\mathcal{O}(\epsilon^{-N}) = \mathcal{O}(\epsilon^q)$$

hence  $\mathbf{AC} \in \mathcal{N}^{(0)}(\mathbb{R})$ . We call the set of elements  $\mathbf{C} \in \mathcal{H}(\mathbb{R})$  satisfying (29), *moderate* and set of moderate functions

$$\mathcal{E}^{(0)}(\mathbb{R}) = \{\mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{there exists } N \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_0, \mathbf{A}[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N})\} \quad (30)$$

Examples of moderate functions include

$$\bar{\Delta}[\phi_\epsilon](y) = \phi_\epsilon(-y) = \frac{1}{\epsilon} \phi\left(-\frac{y}{\epsilon}\right) = \mathcal{O}(\epsilon^{-1}), \quad (\bar{\Delta})^n[\phi_\epsilon] = \mathcal{O}(\epsilon^{-n})$$

and

$$\tilde{f}[\phi_\epsilon](y) = f(y) = \mathcal{O}(\epsilon^0)$$

## 8 Derivatives

The last part in the construction of the Colombeau Algebra is to extend all the definitions so that they also apply to the derivatives  $\frac{d\mathbf{A}[\phi]}{dy}$ ,  $\frac{d^2\mathbf{A}[\phi]}{dy^2}$ , etc. We require that not only does a moderate function not blow up too quickly, but neither do its derivatives, i.e.

$$(\mathbf{A}[\phi])^{(n)} = \frac{d^n}{dy^n}(\mathbf{A}[\phi]) \in \mathcal{E}^{(0)}(\mathbb{R}) \quad (31)$$

Thus we define the set of moderate function as

$$\mathcal{E}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{E}^{(0)}(\mathbb{R}) \mid (\mathbf{A}[\phi])^{(n)} \in \mathcal{E}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N}, \phi \in \mathbb{A}_0 \right\} \quad (32)$$

That is

$$\mathcal{E}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ there exists } N \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_0, \right. \\ \left. (\mathbf{A}[\phi_\epsilon])^{(n)} = \mathcal{O}(\epsilon^{-N}) \right\} \quad (33)$$

Likewise we require that for two generalised functions to be equivalent then we require that all the derivatives are small

$$\mathcal{N}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{N}^{(0)}(\mathbb{R}) \mid (\mathbf{A}[\phi])^{(n)} \in \mathcal{N}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N} \right\} \quad (34)$$

That is

$$\mathcal{N}(\mathbb{R}) = \left\{ \mathbf{A} \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ and } q \in \mathbb{N} \text{ there exists } p \in \mathbb{N} \text{ such that} \right. \\ \left. \text{for all } \phi \in \mathbb{A}_p, (\mathbf{A}[\phi_\epsilon])^{(n)} = \mathcal{O}(\epsilon^q) \right\} \quad (35)$$

## 9 Quotient Algebra

We write the Colombeau Algebra as a quotient algebra,

$$\mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R}) \quad (36)$$

This means that, with regard to elements  $\mathbf{A}, \mathbf{B} \in \mathcal{E}(\mathbb{R})$  we say  $\mathbf{A} \sim \mathbf{B}$  if  $\mathbf{A} - \mathbf{B} \in \mathcal{N}(\mathbb{R})$ . For elements in  $\mathbf{A}, \mathbf{B} \in \mathcal{G}(\mathbb{R})$  we simply write  $\mathbf{A} = \mathbf{B}$ .

Given  $\mathbf{A} \in \mathcal{G}(\mathbb{R})$ , then in order to get an actual number we must first choose a representative  $\mathbf{B} \in \mathcal{E}(\mathbb{R})$  of  $\mathbf{A} \in \mathcal{G}(\mathbb{R})$ , then we must choose  $\phi \in \mathbb{A}_0$  and  $y \in \mathbb{R}$  then the quantity  $\mathbf{B}[\phi](y) \in \mathbb{R}$ .

## 10 Summary

We can summarise the steps needed to go from distributions to Colombeau functions:

- Convert distributions which give a number  $\Psi[\phi]$  as an answer to functionals  $\mathbf{A}[\phi]$  which give a function as an answer.
- Construct the sets of test functions  $\mathbb{A}_q$ , so that  $\bar{f} \sim \tilde{f}$ , i.e.  $\bar{f} - \tilde{f} \in \mathcal{N}^{(0)}(\mathbb{R})$
- Limit the generalised functions to elements of  $\mathcal{E}^{(0)}(\mathbb{R})$  so that the set  $\mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{E}^{(0)}(\mathbb{R})$  is an ideal.
- Extend the definitions of  $\mathcal{E}^{(0)}(\mathbb{R})$  and  $\mathcal{N}^{(0)}(\mathbb{R})$  to  $\mathcal{E}(\mathbb{R})$  and  $\mathcal{N}(\mathbb{R})$  so that they also apply to derivatives.
- Define the Colombeau Algebra as the quotient  $\mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ .

The formal definition, we define  $\mathcal{E}(\mathbb{R})$  via (33), then  $\mathcal{N}(\mathbb{R})$  via (35) and (24). Then define the Colombeau Algebra  $\mathcal{G}(\mathbb{R})$  as the quotient (36).



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